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ON THE ELEMENTARY SYMMETRIC FUNCTIONS OF A SUM OF MATRICES

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Abstract

Often in mathematics, it is useful to summarize a multivariate phenomenon with a single number. In fact, the determinant - which is denoted by det - is one of the simplest cases and many of its properties are very well-known. For instance, the determinant is a multiplicative function, i.e., $\det(AB) = \det A \cdot \det B$, $A, B \in M_n$, and it is a multilinear function, but it is not, in general, an additive function, i.e., $\det(A + B) \neq \det A + \det B$.

Another interesting scalar function in the Matrix Analysis is the characteristic polynomial. In fact, given a square matrix A, the coefficients of its characteristic polynomial $\chi_A(t) := \det(tI - A)$ are, up to a sign, the elementary symmetric functions associated with the eigenvalues of A.

In the present paper, we present new expressions related to the elementary symmetric functions of sum of matrices.

The main motivation of this manuscript is try to find new properties to probe the following conjecture.

Bessis-Moussa-Villani conjecture: [2, 4]

The polynomial $p(t) := Tr((A + tB)^m) \in \mathbb{R}[t]$, has only nonnegative coefficients whenever $A, B \in M_r$ are positive semidefinite matrices.

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Moreover, some numerical evidences and the Newton-Girard formulas suggested to us to consider a more general conjecture that will be considered in a further manuscript.

Positivity Conjecture :

The polynomial $S_k((A + tB)^m) \in \mathbb{R}[t]$, has only nonnegative coefficients whenever $A, B \in M_r$ are positive semidefinite matrices for every $k = 0, 1, \ldots, r$.

It is clear that the BMV conjecture is a particular case of the positivity conjecture for $k = 1$, since $S_1 \equiv Tr$.

1. Introduction

Denote by $M_{m,n}$, the set of $m \times n$ matrices over an arbitrary field $\mathbb{F}$ and by M_n the set $M_{n,n}$. Determinants are mathematical objects that are very useful in the matrix analysis.

In fact, the determinant of a matrix $A \in M_n$, can be presented in two important, apparently different, but equivalent ways.

The first one is the Laplace Expansion:

If $A = [a_{i,j}]$ and assuming that the determinant is defined over M_{n-1}, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}), \tag{1}$$

where $A_{i,j} \in M_{n-1}$ denotes the submatrix of A resulting from the deletion of row i and column j.

The second way is the Alternating Sum:

$$\det(A) = \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}, \tag{2}$$

where P_n is the set of all permutations of $\{1, 2, \ldots, n\}$, and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Remark 1.1. Notice that with these definitions it is clear that the determinant is a multilinear function.

In the present paper, we present a closed expression for

$$\det(A_1 + A_2 + \cdots + A_N), \quad A_1, A_2, \ldots, A_N \in M_n,$$

where $N \geq n+1$, in terms of the sum of another determinants involving the matrices $A_1, A_2, \ldots, A_N$.

Definition 1.2. Let $A \in M_{m,n}$. For any index sets α, β, with $\alpha \subseteq \{1, \ldots, m\}$, $\beta \subseteq \{1, \ldots, n\}$, and $|\alpha| = |\beta|$, we denote the submatrix that lies in the rows of A indexed by α and the columns indexed by β as $A(\alpha, \beta)$.

For example

$$\begin{bmatrix} 1 & 0 & 9 & 0 & -2 \\ 2 & 1 & 7 & 1 & 1 \end{bmatrix}(\{1\}, \{1, 3\}) = [1 \;\; 9].$$

On the other hand and taking into account some properties of the determinant it is well-known that the characteristic polynomial of a given square matrix A can be written as

$$\chi_A(t) = \det(tI - A) = t^n - S_1(A)t^{n-1} + \cdots + (-1)^n S_n(A),$$

where $I \in M_n$ is the identity, and $S_k(A)$ is the elementary symmetric function associated to the matrix A, $k = 1, 2, \ldots, n$.

In fact, by the second way as we have defined the determinant, i.e., the alternating sum, it is straightforward that

$$S_k(A) = \sum_{|\alpha|=k} \det(A(\alpha, \alpha)), \quad k = 1, 2, \ldots, n. \tag{3}$$

In connection with the elementary symmetric functions, we present new equalities related to these functions, giving explicit expressions for $S_2(A+B)$ and $S_3(A+B)$, for any $A, B \in M_n$.

The structure of this paper is the following: In Section 2, we present some results related with the determinant of sum of matrices, whose proof is given in Appendix. In Section 3, we obtain the values of $S_2(A+B)$ and $S_3(A+B)$ by using the definition of the elementary symmetric functions of a matrix, In Section 4, we prove the same identities and also we obtain $S_4(A+B)$ by using the Newton-Girard identities, where A and B are two generic n-by-n matrices.

2. The Determinant of a Sum of Matrices

Let N be a positive integer and let us consider the N-tuple of n-by-n matrices

$$S := (A_1, A_2, \cdots, A_N).$$

We define $\sum(S)$ as the set of all possible formal sums of matrices of S, where each A_i, $i = 1, \ldots, N$, appears at most once.

Remark 2.1. Note that WLG we can add the null matrix, 0, in $\sum(S)$.

The following result will be useful for further results

Theorem 2.2. *Given* $A \in M_n$ *and an integer N, with* $N \geq n+1$. *For any N-tuple* $S = (A_1, A_2, \cdots, A_N)$, $A_i \in M_n$, $i = 1, \ldots, N$, *the following relation holds*:

$$\sum_{k=0}^{N}(-1)^k \sum_{\substack{\Omega\in\sum(S)\\ |\Omega|=k}} \det\left(A + \sum_{A_i\in\Omega} A_i\right) = 0, \tag{4}$$

understanding that $|\Omega| = k$ *means that* Ω *is a formal sum with k summands, and that* $A_i \in \Omega$ *means that* A_i *is a summand in* Ω.

Remark 2.3. The identity (4) can be rewritten as

$$\sum_{x_1,\ldots,x_N=0}^{1} (-1)^{x_1+\cdots+x_N} \det\left(A + \sum_{j=1}^{N} x_j A_j\right) = 0. \tag{5}$$

Chapman proves in [7] the case $A = 0$ of this. But his argument works as well in this generalized form; the determinant is a polynomial of degree less than N in the variables $x_1, \ldots, x_N$ and this alternating sum must vanish as seen by applying to any monomial of degree less than N. Alternatively (5) follows by subtracting the $N+1$ case of Chapmans identity from the N case.

For instance, if we set $A = 0$ in (4) and $A_1, A_2, A_3, A_4 \in M_3$, i.e., $N = 4$, then

$$\begin{aligned}\det(A_1 + A_2 + A_3 + A_4) = {} & \det(A_1 + A_2 + A_3) + \det(A_1 + A_2 + A_4) \\ & + \det(A_1 + A_3 + A_4) + \det(A_2 + A_3 + A_4) \\ & - \det(A_1 + A_2) - \det(A_1 + A_3) - \det(A_1 + A_4) \\ & - \det(A_2 + A_3) - \det(A_2 + A_4) - \det(A_3 + A_4) \\ & + \det(A_1) + \det(A_2) + \det(A_3) + \det(A_4).\end{aligned}$$

This result has very interesting consequences.

Corollary 2.4. *Under the conditions of Theorem* 2.2. *For any index sets* $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ *of size* τ, $N \geq \tau + 1$, *the following relation holds*:

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det\left(A(\alpha, \beta) + \sum_{A_i \in \Omega} A_i(\alpha, \beta) \right) = 0. \tag{6}$$

The proof follows from Theorem 2.2 replacing A by $A(\alpha, \beta)$ and taking into account that $A_i(\alpha, \beta) \in M_\tau$ and $N \geq \tau + 1$.

On the other hand, if we combine the above result and (3) we obtain that:

Corollary 2.5. *Under the conditions of Theorem* 2.2, *for any nonnegative integer* τ, $N \geq \tau + 1$,

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} S_\tau\left(A + \sum_{A_i \in \Omega} A_i \right) = 0, \tag{7}$$

where $S_\tau(C)$ *is the* τ-*th elementary symmetric function of the matrix C.*

The proof, again, is straightforward taking into account (3) and that $N \geq \tau + 1$.

The following identity is useful to compute τ-th elementary symmetric function of any number of matrices $N \geq \tau + 1$.

Corollary 2.6. *Under the conditions of Theorem* 2.2. *For any nonnegative integers* τ, $N \geq \tau + 1$, *the following identity fulfills*

$$S_{\tau}(A_1 + A_2 + \cdots + A_N) = \sum_{j=0}^{k-1} (-1)^j \binom{j + N - k - 1}{N - k - 1} \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k-j}} S_{\tau}\left(\sum_{i \in \Omega} A_i\right), \quad (8)$$

The proof is elementary and we leave it for the reader.

In fact, Theorem 2.2 is optimal with respect to the range of *N*, i.e., for every positive integer *n*, it is possible to find *n*-tuples of M_n such that the equality (4), given in Theorem 2.2, fails. For instance, taking

$$A_i = \operatorname{diag}(e_i), \quad i = 1, 2, \ldots, n, \quad A = xe_1, \; x \in \mathbb{R},$$

where $\{e_1, e_2, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, it is straightforward to check that

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det\left(A + \sum_{A_i \in \Omega} A_i\right) = (-1)^n (1 + x - x) \neq 0.$$

3. Obtaining $S_2(A + B)$ and $S_3(A + B)$

So the next logical step is to get closed expressions for the τ-th elementary symmetric functions of a sum of *N* matrices, with $1 \leq N \leq \tau$. To do that we will use the Newton-Girard formulas for the elementary symmetric functions (see, e.g., [5, Subsection 10.12]) and the definition of such functions (3).

Remark 3.1. Note that if *A* is *n*-by-*n*, then $\det(A) = S_n(A)$, so it is enough to obtain those identities for the elementary symmetric functions and then apply these to the determinant.

Lemma 3.2. *For any* $A_1, A_2 \in M_n$, *we get*

$$S_2(A_1 + A_2) = S_2(A_1) + S_2(A_2) + S_1(A_1)S_1(A_2) - S_1(A_1A_2), \tag{9}$$

$$\begin{aligned} S_3(A_1 + A_2) = {} & S_3(A_1) + S_3(A_2) - S_1(A_1 + A_2)S_1(A_1A_2) \\ & + S_1(A_1)S_2(A_2) + S_1(A_2)S_2(A_1) + S_1(A_1^2A_2) \\ & + S_1(A_1A_2^2). \end{aligned} \tag{10}$$

$$\begin{aligned} S_3(A_1 + A_2 + A_3) = {} & S_3(A_1) + S_3(A_2) + S_3(A_3) - S_1(A_1 + A_2 + A_3) \\ & \times S_1(A_1A_2 + A_1A_3 + A_2A_3) + S_1(A_1)(S_2(A_2) \\ & + S_2(A_3)) + S_1(A_2)(S_2(A_1) + S_2(A_3)) + S_1(A_3) \\ & \times (S_2(A_1) + S_2(A_2)) + S_1(A_1^2A_2) + S_1(A_1A_2^2) \\ & + S_1(A_1^2A_3) + S_1(A_1A_3^2) + S_1(A_2^2A_3) + S_1(A_2A_3^2) \\ & + S_1(A_1A_2A_3) + S_1(A_1A_3A_2). \end{aligned} \tag{11}$$

Proof. Let $A_1, A_2 \in M_n$ be two matrices with spectra $\sigma(A_1) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(A_2) = \{\mu_1, \ldots, \mu_n\}$, respectively. The Newton-Girard formula gives

$$\sum_{i=1}^{n} \lambda_i^2 = S_1^2(A_1) - 2S_2(A_1),$$

where $S_1(A_1) \equiv Tr(A_1)$. WLG we can assume A_1 diagonal, then by definition of S_2 (see (3)) we get

$$\begin{aligned} S_2(A_1 + A_2) &= \sum_{1 \le i < j \le n} \lambda_i\lambda_j + 2\sum_{j=1}^{n} \lambda_j(Tr(A_2) - b_{j,j}) + S_2(A_2) \\ &= \frac{1}{2}\left(\sum_{j=1}^{n} \lambda_j\right)^2 - \frac{1}{2}\sum_{j=1}^{n} \lambda_j^2 + 2Tr(A_1)Tr(A_2) - 2Tr(A_1A_2) \end{aligned}$$

$$+ S_2(A_2) = \frac{1}{2}(Tr(A_1))^2 - \frac{1}{2}\left((Tr(A_1))^2 - 2S_2(A_1)\right)$$

$$+2Tr(A_1)Tr(A_2) - 2Tr(A_1A_2) + S_2(A_2)$$

$$= S_2(A_1) + S_2(A_2) + Tr(A_1)Tr(A_2) - Tr(A_1A_2).$$

And for $S_3(A_1 + A_2)$, if A_1 is a diagonal matrix, by definition of S_3 (see (3)), it is straightforward to get

$$S_3(A_1 + A_2) = \sum_{1\le i<j<k\le n} \lambda_i\lambda_j\lambda_k + \sum_{1\le i<j\le n} \lambda_i\lambda_j\left(Tr(A_2) - b_{i,i} - b_{j,j}\right)$$

$$+ \sum_{j=1}^{n} \lambda_j S_2((A_2)_{j,j}) + S_3(A_2).$$

Taking into account that in this case the Newton-Girard formula produces the identity

$$\sum_{i=1}^{n} \lambda_i^3 = S_1^3(A_1) - 3S_1(A_1)S_2(A_1) + 3S_3(A_1),$$

and the expansion of $(a + b + c)^3$, we obtain

$$S_3(A_1 + A_2) = \frac{1}{6}((Tr(A_1))^3 + 2Tr(A_1^3) - 3Tr(A_1^2)Tr(A_1))$$

$$+ S_2(A_1)Tr(A_2) - Tr(A_1)Tr(A_1A_2) + Tr(A_1^2A_2)$$

$$+ \sum_{j=1}^{n} \lambda_j S_2((A_2)_{j,j}) + S_3(A_2).$$

But we can assume A_2 is a diagonal matrix and say

$$S_2((A_2)_{j,j}) = S_2(A_2) - \mu_j Tr(A_2) + \mu_j^2, \quad j = 1, 2, \ldots, n.$$

So,

$$S_3(A_1 + A_2) = S_3(A_1) + S_2(A_1)Tr(A_2) - Tr(A_1)Tr(A_1A_2)$$

$$+ Tr(A_1^2 A_2) + 3Tr(A_1)S_2(A_2) - Tr(A_1A_2)Tr(A_2)$$

$$+ Tr(A_1A_2^2) + S_3(A_2),$$

and hence both relations, (9) and (10), hold. Moreover (11) is a direct consequence of (10). □

4. Other Way to Obtain $S_2(A+B)$, $S_3(A+B)$ and $S_4(A+B)$

We will start probing $S_2(A+B)$ using the Newton-Girard identities:

$$-2S_2(A+B) = S_1((A+B)^2) - S_1^2(A+B) = -S_1^2(A) + S_1(A^2) - S_1^2(B)$$

$$+ S_1(B^2) - 2S_1(A)S_1(B) + 2S_1(AB) = -2S_2(A) - 2S_2(B)$$

$$-2S_1(A)S_1(B) + 2S_1(AB).$$

We will apply an analogous way to obtain $S_3(A+B)$:

$$3S_3(A+B) = S_1((A+B)^3) - S_1((A+B)^2)S_1(A+B) + S_1(A+B)S_2(A+B)$$

$$= S_1^3(A) + 3S_1(A^2B) + 3S_1(AB^2) + S_1(B^3) - S_1(A^2)$$

$$\times S_1(A) - S_1(A^2)S_1(B) - 2S_1(AB)S_1(A) - 2S_1(AB)S_1(B)$$

$$- S_1(B^2)S_1(A) - 2S_1(B^2)S_1(B) + S_1(A+B)S_2(A+B).$$

If now we expand $S_2(A+B)$, after some simplifications it is clear we get the desired identity for $S_3(A+B)$.

4.1. Obtaining $S_4(A+B)$

As the above examples, the Newton-Girard formula gives

$$-4S_4(A+B) = S_1((A+B)^4) - S_1((A+B)^3)S_1(A+B) + S_1((A+B)^2)$$

$$\times S_2(A+B) - S_1(A+B)S_3(A+B).$$

Taking into account the properties of the trace, we get

$$\begin{aligned}
-4S_4(A+B) = {} & S_1(A^4) + 4S_1(A^3B) + 4S_1(A^2B^2) + 2S_1((AB)^2) + 4S_1(AB^3) \\
& + S_1(B^4) - S_1(A^3)S_1(A) - S_1(A^3)S_1(B) - 3S_1(A^2B)S_1(A) \\
& - 3S_1(A^2B)S_1(B) - 3S_1(AB^2)S_1(A) - 3S_1(AB^2)S_1(B) \\
& - S_1(B^3)S_1(A) - S_1(B^3)S_1(B) + (S_1(A^2) + 2S_1(AB) \\
& + S_1(B^2))(S_2(A) + S_2(B) + S_1(A)S_1(B) - S_1(AB)) \\
& -(S_1(A+B))(S_3(A) + S_3(B) + S_1(A)S_2(B) + S_1(B)S_2(A) \\
& + S_1(A^2B) + S_1(AB^2) - S_1(AB)S_1(A) - S_1(AB)S_1(B)).
\end{aligned}$$

Applying the same technique applied before, we get

$$\begin{aligned}
-4S_4(A+B) = {} & -4S_4(A) - 4S_4(B) \\
& 4S_1(A^3B) + 4S_1(A^2B^2) + 2S_1((AB)^2) + 4S_1(AB^3) \\
& - S_1(A^3)S_1(B) - 3S_1(A^2B)S_1(A) - 3S_1(A^2B)S_1(B) \\
& - 3S_1(AB^2)S_1(A) - 3S_1(AB^2)S_1(B) - S_1(B^3)S_1(A) \\
& + S_1(A^2)(S_2(B) + S_1(A)S_1(B) - S_1(AB)) + 2S_1(AB)(S_2(A) \\
& + S_2(B) + S_1(A)S_1(B) - S_1(AB)) + S_1(B^2)(S_2(A) \\
& + S_1(A)S_1(B) - S_1(AB)) - S_1(A)(S_3(B) + S_1(A)S_2(B) \\
& + S_1(B)S_2(A) + S_1(A^2B) + S_1(AB^2) \\
& - S_1(AB)S_1(A) - S_1(AB)S_1(B)) - S_1(B)(S_3(A) \\
& + S_1(A)S_2(B) + S_1(B)S_2(A) + S_1(A^2B) + S_1(AB^2) \\
& - S_1(AB)S_1(A) - S_1(AB)S_1(B)).
\end{aligned}$$

After some simplifications applying the Newton-Girard formulas, we get

$$4S_4(A+B) = 4S_1(A) + 4S_4(B) - 4S_1(A^3B) - 4S_1(A^2B^2) - 2S_1((AB)^2)$$

$$- 4S_1(AB^3) + 3S_1(A^2B)S_1(A) + 3S_1(A^2B)S_1(B) + 3S_1(AB^2)$$

$$\times S_1(A) + 3S_1(AB^2)S_1(B) - 2S_1(AB)(S_1(A)S_1(B)$$

$$+ S_1(AB)) + S_1(A)(S_1(A^2B) - S_1(AB^2) + S_1(AB)S_1(B))$$

$$+ S_1(B)(S_1(A^2B) - S_1(AB^2) + S_1(AB)S_1(A))$$

$$+4S_3(A)S_1(B) + 4S_3(B)S_1(A) + 4S_2(A)S_2(B)$$

$$-4S_2(A)S_1(AB) - 4S_2(B)S_1(AB).$$

Applying the Newton-Girard formulas and after some simplifications, we get

$$S_4(A + B) = S_4(A) + S_4(B) - S_1(A^3B) - S_1(A^2B^2) - S_1(AB^3) + S_1(A^2B)$$

$$\times S_1(A) + S_1(A^2B)S_1(B) + S_1(AB^2)S_1(A) + S_1(AB^2)S_1(B)$$

$$-S_1(AB)S_1(A)S_1(B) + S_3(A)S_1(B) + S_3(B)S_1(A) + S_2(A)$$

$$\times S_2(B) - S_2(A)S_1(AB) - S_2(B)S_1(AB) + S_2(AB).$$

5. Conclusions and Outlook

We have constructed the 2nd, the 3rd and the 4th elementary symmetric functions of a sum of two matrices but, of course, is simple to see that is possible to compute the τ-th elementary symmetric function of a sum of N-matrices, $1 \leq N \leq \tau$ by using the Newton-Girard formulas or by using the same technique used in Lemma 3.2 which, by the way, is too much complicated.

Of course, one of the goals in further papers is to find a closed expression in the general case which for the moment is not clear although we believe the Theory of partition of integers is involved.

In fact, by using the generalized Waring's formula [6], for any A is n-by-n matrix, $0 \leq k \leq n$ and any nonnegative integer m, we get

$$S_k(A^m) = (-1)^{k(n+1)} \sum_{|\lambda|=kn} A_\lambda e_\lambda(A),$$

where the coefficients A_λ are given by

$$A_\lambda = \sum_{\substack{|\pi|=k \\ \pi=(k_1,k_2,\dots)}} \sum_{\substack{\lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_{l(\pi)} = \lambda \\ |\lambda_i| = k_i n}} \frac{(-1)^{l(\lambda)-l(\pi)}}{\prod_{i=1}^{k} m_i(\pi)!} \prod_{l=1}^{l(\pi)} \frac{n}{l(\lambda_i)} \binom{l(\lambda_i)}{m_1(\lambda_i), \dots, m_m(\lambda_i)},$$

and $e_\lambda(A) = S_1^{m_1(\lambda)}(A) S_2^{m_2(\lambda)}(A) \cdots$.

Remark 5.1. A partition is a finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers in decreasing order, where $l(\lambda)$ denotes the length of the partition, and $m_k(\lambda)$ denotes the number of parts of λ equal to k.

$\lambda \cup \mu$ is the partition whose parts are those of λ and μ.

Taking into account this identity, we believe that we can obtain an analogous expression for the τ-th elementary symmetric function of a sum of matrices. In fact, we expect one expression in which appears the elementary symmetric functions on words of the letters $\mathrm{A}_1, A_2, \dots, A_N$ as one could see in Subsection 4.1 for the case $S_4(A_1 + A_2)$.

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Appendix

Proof of Theorem 2.2

We will prove by induction on n:

• If $n = 1$ the matrices are scalars so, for every k,

$$\sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega|=k}} \det\left(A + \sum\nolimits_{A_i \in \Omega} A_i\right) = \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega|=k}} \left(A + \sum_{A_i \in \Omega} A_i\right)$$

$$= \binom{N}{k} A + \binom{N-1}{k-1} (A_1 + A_2 + \cdots + A_N),$$

and hence (4) holds for $n = 1$ and $N \geq 2$.

• If we assume that the result holds for n, let us going to prove the identity (4) for $n + 1$. Taking the Laplace expansion through the first row, we get

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det(A + \sum_{A_i \in \Omega} A_i) = \sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \{ \sum_{j=1}^{n+1} (-1)^{1+j}$$

$$\times (A(1, j) + \sum_{A_i \in \Omega} A_i(1, j)) \det(A_{1,j} + \sum_{A_i \in \Omega} (A_i)_{1,j}) \}.$$

By induction, since for every $j = 1, 2, \ldots, n$, $A(1, j)$ is fixed and does not depend on k nor Ω, we get that the above expression is equal to

$$\sum_{j=1}^{n+1} (-1)^{j+1} \sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \sum_{A_i \in \Omega} A_i(1, j)) \det(A_{1,j} + \sum_{A_i \in \Omega} (A_i)_{1,j})$$

$$= \sum_{j=1}^{n+1} (-1)^{j+1} \sum_{k=0}^{N} (-1)^k \sum_{\lambda=1}^{N} A_\lambda(1, j) \sum_{\substack{\Omega \in \Sigma(S) \\ A_\lambda \in \Omega \\ |\Omega| = k}} \det(A_{1,j} + \sum_{A_i \in \Omega} (A_i)_{1,j}).$$

Now, if we assume that any set with less than one element has determinant equal to zero, we get

$$\sum_{j=1}^{n+1} (-1)^{j+1} \sum_{k=0}^{N} (-1)^k \sum_{\lambda=1}^{N} A_\lambda(1, j) \sum_{\substack{\tilde{\Omega} \in \Sigma(S \setminus \{A_\lambda\}) \\ |\tilde{\Omega}| = k-1}} \det(A_{1,j} + (A_\lambda)_{1,j} + \sum_{A_i \in \tilde{\Omega}} (A_i)_{1,j})$$

$$= \sum_{j=1}^{n+1} (-1)^{j+1} \sum_{\lambda=1}^{N} A_\lambda(1, j) \sum_{k=0}^{N-1} (-1)^k \sum_{\substack{\tilde{\Omega} \in \Sigma(S \setminus \{A_\lambda\}) \\ |\tilde{\Omega}| = k}} \det(A_{1,j} + (A_\lambda)_{1,j} + \sum_{A_i \in \tilde{\Omega}} (A_i)_{1,j}).$$

By induction, since for every λ and j, the matrices $A_{1,j}, (A_\lambda)_{1,j} \in M_{n-1}$ are fixed, thus $A = A_{1,j} + (A_\lambda)_{1,j} \in M_{n-1}$ is also fixed. Thus we get

$$\sum_{j=1}^{n+1} (-1)^{j+1} \sum_{\lambda=1}^{N} A_\lambda(1, j) \cdot 0 = 0.$$

Moreover, since $N \geq n+2$, then $N - 1 \geq n + 1$. Hence, the relation holds. □

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